8.2 - Homogeneous Linear Systems

Consider the system $\begin{array}{l}\dfrac{dx}{dt} = 2x + 2y \\ \dfrac{dy}{dt} = x + 3y\end{array}$ which, in $\mathbf{X}' = \mathbf{A}\mathbf{X}$ form is

$\begin{pmatrix} x' \\ y' \end{pmatrix}$ $\begin{pmatrix} x \\ y \end{pmatrix}$

$\mathbf{X}' = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \mathbf{X}$.

The solution is $\vec{X}(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{4t} + c_2\begin{pmatrix}2\\-1\end{pmatrix}e^{t}$

This is a superposition of

$\vec{X}_1(t) = \begin{pmatrix}1\\1\end{pmatrix}e^{4t}$, $\vec{X}_2(t) = \begin{pmatrix}2\\-1\end{pmatrix}e^{t}$

These have the form $\vec{X}(t) = \vec{K}e^{\lambda}$

λ is the solution to an equation;

it is called an eigenvalue.

$\vec{K}$ is the associated eigenvector. cannot be $\vec{0}$

Suppose $\vec{X} = \vec{K}e^{\lambda t}$ is a solution

to $\vec{X}' = A\vec{X}$.

Substitute: $\vec{X}' = \lambda\vec{K}e^{\lambda t} \Rightarrow \lambda\vec{K}e^{\lambda t} = A\vec{K}e^{\lambda t}$

$\lambda\vec{K} = A\vec{K} \Rightarrow A\vec{K} - \lambda\vec{K} = \vec{0}$ $\quad e^{\lambda t} \neq 0$

$(A - \lambda I)\vec{K} = \vec{0}$ $\quad \vec{K} = \vec{0}$ either $\begin{pmatrix}0\\0\end{pmatrix}$ or $\begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Downarrow$ is the trivial solution

I is the identity matrix: $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ or $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

From linear algebra:

λ is an eigenvalue of A (with associated eigenvector $\vec{K}$) if

$$\det(A-\lambda I)=0$$

characteristic equation

This is associated with a dependent system of equations.

Example: Find the general solution of the given system.

$$\frac{dx}{dt}=2x+2y$$
$$\frac{dy}{dt}=x+3y$$

$$A=\begin{pmatrix}2&2\\1&3\end{pmatrix} \Rightarrow A-\lambda I=\begin{pmatrix}2&2\\1&3\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}$$

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix}2-\lambda&2\\1&3-\lambda\end{vmatrix}=0$$

$$(2-\lambda)(3-\lambda)-2=0$$

$$\lambda^2-5\lambda+6-2=0 \Rightarrow \lambda^2-5\lambda+4=0$$

$$(\lambda-4)(\lambda-1)=0 \Rightarrow \lambda=1,4$$

$\lambda_1=1$ into $(A-\lambda I)\vec{K}=\vec{0}$

$$\begin{pmatrix}1&2\\1&2\end{pmatrix}\vec{K}_1=\vec{0}$$

$\left(\text{This is } \begin{pmatrix}1&2\\1&2\end{pmatrix}\begin{pmatrix}k_1\\k_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\right.$

$k_1+2k_2=0$
$k_1+2k_2=0$ $\Big)$

Such a system can be solved using an augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right)$ $R_2 \to R_1 - R_2$

we want 0 here

$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \to k_1 + 2k_2 = 0$

← dependent

$\Rightarrow k_1 = -2k_2$ Let $k_2 = -1$.

Then $k_1 = 2$ and $\vec{K}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda_2 = 4$ $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \vec{K}_2 = \vec{0} \Rightarrow \cancel{\left(\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right)}$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \xrightarrow[R_2 \to R_1 + 2R_2]{} \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \xrightarrow[R_1 \to \frac{1}{2}R_1]{} \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow k_1 = k_2$$

Let $k_2 = 1$. Then $k_1 = 1$ and $\vec{K}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{X}_1(t) = \vec{K}_1 e^{\lambda_1 t}$, $\vec{X}_2(t) = \vec{K}_2 e^{\lambda_2 t}$

Here, $\vec{X}_1(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t}$, $\vec{X}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

and $\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}$

Recap: $\vec{X}' = A\vec{X} \Rightarrow \det(A - \lambda I) = 0$ to find λ

$\to (A - \lambda I)\vec{K} = \vec{0}$ to find $\vec{K}$

$\vec{X}_i = \vec{K}_i e^{\lambda_i t}$ Then $\vec{X}(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t} + \cdots$

$$\begin{pmatrix} 2 & -2 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 2x - 7y$$
$$\frac{dy}{dt} = 5x + 10y + 4z$$
$$\frac{dz}{dt} = 5y + 2z$$

$$\begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 10-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} + 7\begin{vmatrix} 5 & 4 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(10-\lambda)(2-\lambda)-20\right] + 7\cdot 5(2-\lambda) = 0$$

$$(2-\lambda)\left(\cancel{20}-12\lambda+\lambda^2-\cancel{20}+35\right) = 0$$

$$(2-\lambda)(\lambda^2-12\lambda+35) = 0 \Rightarrow (2-\lambda)(\lambda-5)(\lambda-7) = 0$$

$$\lambda = 2, 5, 7 \quad \to \quad (A-\lambda I)\vec{K} = \vec{0}$$

$\lambda_1 = 2$
$$\begin{pmatrix} 0 & -7 & 0 \\ 5 & 8 & 4 \\ 0 & 5 & 0 \end{pmatrix}\vec{K}_1 = \vec{0}$$

$R_3 \to 5R_1 + 7R_3$

Then $R_1 \to -\frac{1}{7}R_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 5 & 8 & 4 \\ 0 & 0 & 0 \end{pmatrix} \quad R_2 \to R_2 - 8R_1 \quad \to \begin{pmatrix} 0 & 1 & 0 \\ 5 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_1: k_2 = 0$. $\quad R_2: 5k_1 + 4k_3 = 0$

$$k_1 = -\frac{4}{5}k_3$$

$$\vec{K}_1 = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} \to \vec{X}_1(t) = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} e^{2t}$$

$\det(A - \lambda I) = 0$

$\lambda_2 = 5$ and we had $\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 0$

$\hookrightarrow (A - \lambda I)\vec{K} = \vec{0} \rightarrow \begin{pmatrix} -3 & -7 & 0 \\ 5 & 5 & 4 \\ 0 & 5 & -3 \end{pmatrix}$

$R_2 \Rightarrow 5R_1 + 3R_2$

$$\begin{array}{ccc} -15 & -35 & 0 \\ 15 & 15 & 12 \\ \hline 0 & -20 & 12 \end{array}$$

$\begin{pmatrix} -3 & -7 & 0 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \Rightarrow -3k_1 - 7k_2 = 0 \\ \Rightarrow 5k_2 - 3k_3 = 0 \end{array} \Rightarrow \begin{array}{l} k_1 = -\frac{7}{3}k_2 \\ k_3 = \frac{5}{3}k_2 \end{array}$

Let $k_2 = 3$ Then $\vec{K}_2 = \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix}$.

$\lambda_3 = 7 \quad \begin{pmatrix} -5 & -7 & 0 \\ 5 & 3 & 4 \\ 0 & 5 & -5 \end{pmatrix}$ $\quad R_2 \Rightarrow R_1 + R_2$

$$\begin{array}{ccc} -5 & -7 & 0 \\ 5 & 3 & 4 \\ \hline 0 & -4 & 4 \end{array}$$

Note: This is equivalent to R_3

$\begin{pmatrix} 5 & 7 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{array}{l} k_1 = -\frac{7}{5}k_2 \\ k_3 = k_2 \end{array} \Rightarrow \vec{K}_3 = \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix}$

$$\boxed{\vec{X}(t) = c_1 \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -7 \\ 3 \\ 5 \end{pmatrix} e^{5t} + c_3 \begin{pmatrix} -7 \\ 5 \\ 5 \end{pmatrix} e^{7t}}$$

meaning:

$$\begin{aligned} x(t) &= 4e^{2t} \cdot c_1 - 7e^{5t} \cdot c_2 - 7e^{7t} \cdot c_3 \\ y(t) &= \qquad\quad\; 3e^{5t} \cdot c_2 + 5e^{7t} \cdot c_3 \\ z(t) &= -5e^{2t} \cdot c_1 + 5e^{5t} \cdot c_2 + 5e^{7t} \cdot c_3 \end{aligned}$$

becomes $\underbrace{\begin{bmatrix} 4e^{2t} & -7e^{5t} & -7e^{7t} \\ 0 & 3e^{5t} & 5e^{7t} \\ -5e^{2t} & 5e^{5t} & 5e^{7t} \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\vec{C}}$

If $\lambda = \alpha + \beta i$, $\bar{\lambda} = \alpha - \beta i$, then we have

$$\vec{X} = \vec{K}e^{\lambda t} \quad \text{and} \quad \vec{X} = \bar{\vec{K}}e^{\bar{\lambda} t}$$

But $\vec{K}e^{(\alpha+\beta i)t} = \vec{K}e^{\alpha t}(\cos\beta t + i\sin\beta t)$

$$\bar{\vec{K}}e^{(\alpha-\beta i)t} = \bar{\vec{K}}e^{\alpha t}(\cos\beta t - i\sin\beta t)$$

By the superposition principle,

$$\vec{X}_1 = \frac{1}{2}\left(\vec{K}e^{\lambda t} + \bar{\vec{K}}e^{\bar{\lambda} t}\right) = \frac{1}{2}\left(\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\cos\beta t$$

$$- \frac{i}{2}\left(-\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\sin\beta t$$

$$\vec{X}_2 = \frac{i}{2}\left(-\vec{K}e^{\lambda t} + \bar{\vec{K}}e^{\bar{\lambda} t}\right) = \frac{i}{2}\left(-\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\cos\beta t + \frac{1}{2}\left(\vec{K}+\bar{\vec{K}}\right)e^{\alpha t}\sin\beta t$$

Aside: $\frac{1}{2}(\lambda + \bar{\lambda}) = \frac{1}{2}(\alpha + \beta i + \alpha - \beta i) = \alpha$ (Real part)

$\frac{i}{2}(-\lambda + \bar{\lambda}) = \frac{i}{2}(-\alpha - \beta i + \alpha - \beta i) = \beta$ (imaginary part)

So we let $\vec{B}_1 = \frac{1}{2}\left(\vec{K}+\bar{\vec{K}}\right) = \mathrm{Re}(\vec{K})$

and $\vec{B}_2 = \frac{i}{2}\left(-\vec{K}+\bar{\vec{K}}\right) = \mathrm{Im}(\vec{K})$

this leads to a theorem:

Theorem: Let $\lambda = \alpha + i\beta$ be an eigenvalue of $\mathbf{A}$ and let $\mathbf{B}_1 = \mathrm{Re}(\mathbf{K})$, $\mathbf{B}_2 = \mathrm{Im}(\mathbf{K})$. Then

$$\mathbf{X}_1 = e^{\alpha t}(\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t)$$
$$\mathbf{X}_2 = e^{\alpha t}(\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t)$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 4x + 5y$$
$$\frac{dy}{dt} = -2x + 6y$$

$$\begin{vmatrix} 4-\lambda & 5 \\ -2 & 6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 10\lambda + 34 = 0$$

$$\lambda^2 - 10 + 25 = -34 + 25$$

$$(\lambda-5)^2 = -9 \Rightarrow \lambda = 5 \pm 3i$$

$$(A - \lambda I)\vec{K} = \vec{0} \Rightarrow \begin{pmatrix} 4-(5+3i) & 5 \\ -2 & 6-(5+3i) \end{pmatrix}\vec{K} = \vec{0}$$

$$\begin{pmatrix} -1-3i & 5 \\ -2 & 1-3i \end{pmatrix}\vec{K} = \vec{0}$$

verify determinant $=0$ or row reduce

R_2: $-2k_1 + (1-3i)k_2 = 0$

$$k_1 = \frac{1-3i}{2}k_2$$ Let $k_2 = 2$. Then $k_1 = 1-3i$

$$\vec{K} = \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\vec{B}_1} + \underbrace{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}_{\vec{B}_2} i$$

$$\vec{X}(t) = c_1 e^{5t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos 3t - \begin{pmatrix} -3 \\ 0 \end{pmatrix}\sin 3t\right] + c_2 e^{5t}\left[\begin{pmatrix} -3 \\ 0 \end{pmatrix}\cos 3t + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin 3t\right]$$

Meaning:

$$\vec{X}(t) = c_1 e^{5t} \begin{pmatrix} \cos 3t + 3\sin 3t \\ 2\cos 3t \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} -3\cos 3t + \sin 3t \\ 2\sin 3t \end{pmatrix}$$

$$\vec{X}(t) = e^{5t} \begin{pmatrix} \cos 3t + 3\sin 3t & -3\cos 3t + \sin 3t \\ 2\cos 3t & 2\sin 3t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Repeated eigenvalues:

Example: Find the general solution of the given system.

$$X' = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} X \qquad \begin{vmatrix} 12-\lambda & -9 \\ 4 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda-6)^2 = 0 \Rightarrow \lambda = 6 \ \text{(multiplicity 2)}$$

$$(A-\lambda I)\vec{K} = \vec{0} \Rightarrow \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{K} = \vec{0}$$

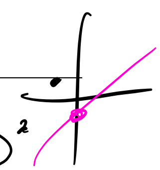
$$\rightarrow \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} \rightarrow 2k_1 = 3k_2$$

$$k_1 = \frac{3}{2}k_2$$

$$\vec{K} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \vec{X}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t}$$

We have only one eigenvector. In this case, we find a vector $\vec{P}$ such that

$$(A-\lambda I)\vec{P} = \vec{K} : \begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix} \vec{P} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



$$\left(\begin{array}{cc|c} 6 & -9 & 3 \\ 4 & -6 & 2 \end{array}\right) \rightarrow \boxed{2p_1 - 3p_2 = 1} \quad \rightarrow 2x - 3y = 1$$

Let $p_2 = 0$.

$p_1 = 5$: $10 - 3p_2 = 1 \Rightarrow p_2 = 3$

Then $p_1 = \frac{1}{2}$. Then $\vec{P} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$

$$\vec{X}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{6t} + c_2 \left[\begin{pmatrix} 3 \\ 2 \end{pmatrix} t e^{6t} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} e^{6t} \right]$$

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

$$c_1 \vec{K} e^{\lambda t} + c_2 \left(\vec{K} t e^{\lambda t} + \vec{P} e^{\lambda t} \right)$$

Rationale for this will come later.

$$\left[c_1 \vec{K} + c_2(\vec{K}t + \vec{P}) + c_3\left(\vec{K}\tfrac{1}{2}t^2 + \vec{P}t + \vec{Q}\right) \right] e^{\lambda t}$$

Example: Find the general solution of the given system.

$$\frac{dx}{dt} = 3x + 2y + 4z$$
$$\frac{dy}{dt} = 2x + 2z$$
$$\frac{dz}{dt} = 4x + 2y + 3z$$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix} = 0$$

$$(3-\lambda)(\lambda^2-3\lambda-4) - 2(6-2\lambda-8) + 4(4+4\lambda) = 0$$

$$(3-\lambda)(\lambda+1)(\lambda-4) + 4(\lambda+1) + 16(\lambda+1) = 0$$

$$(\lambda+1)\left[\overbrace{(3-\lambda)(\lambda-4)}^{-\lambda^2+7\lambda-12} + 20\right] = 0 \Rightarrow -(\lambda+1)(\lambda+1)(\lambda-8) = 0$$

$\lambda = 8, \ -1$ (mult. 2) $\qquad \lambda_1 = 8 \rightarrow \vec{K}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$\lambda_2 = -1 : \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2 rows of 0 will give 2 eigenvectors.

$$2k_1 + k_2 + 2k_3 = 0$$

Let $k_2 = 0$. Then $\vec{K}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Let $k_3 = 0$. Then $2k_1 + k_2 = 0 \Rightarrow k_1 = -\frac{1}{2}k_2$

$$\vec{K}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\boxed{\vec{X}(t) = c_1\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}e^{8t} + c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}e^{-t}}$$

Aforementioned rationale:

In the case of a repeated eigenvalue (2x2 or 3x3)
we have but not enough eigenvectors

$\vec{X}_1 = \vec{K}e^{\lambda t}$] if mult. 2

$\vec{X}_2 = \vec{K}te^{\lambda t} + \vec{P}e^{\lambda t}$

$\vec{X}_3 = \vec{K}\frac{1}{2}t^2 e^{\lambda t} + \vec{P}te^{\lambda t} + Qe^{\lambda t}$] if mult. 3

Suppose $\vec{X} = \vec{K}te^{\lambda t} + \vec{P}e^{\lambda t}$.

Then $\vec{X}' = \vec{K}e^{\lambda t} + \vec{K}\lambda te^{\lambda t} + \vec{P}\lambda e^{\lambda t}$

So $\vec{X}' = A\vec{X}$ becomes

$\vec{K}e^{\lambda t} + \vec{K}\lambda te^{\lambda t} + \vec{P}\lambda e^{\lambda t} = A(\vec{K}te^{\lambda t} + \vec{P}e^{\lambda t})$

$A\vec{K} = \vec{K}\lambda \Rightarrow A\vec{K} - \lambda\vec{K} = \vec{0} \Rightarrow (A - \lambda I)\vec{K} = \vec{0}$

$\vec{K} + \vec{P}\lambda = A\vec{P} \Rightarrow A\vec{P} - \lambda\vec{P} = \vec{K}$

$(A - \lambda I)\vec{P} = \vec{K}$

If $\vec{Q}$ is needed, we use $(A - \lambda I)\vec{Q} = \vec{P}$

2x2 – 1 eigenvalue of mult. 2 $\rightarrow$ get $\vec{K}$ $\rightarrow$ get $\vec{P}$.

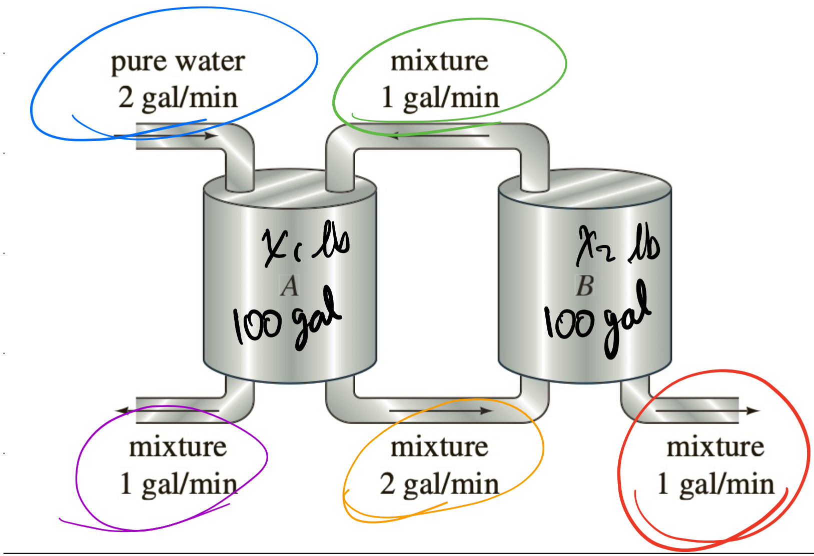
3x3 – • 3 eigenvalues $\rightarrow$ 3 eigenvectors $\vec{K}_1, \vec{K}_2, \vec{K}_3$

I think • λ_1 mult. 2 $\rightarrow$ 2 rows of zeros $\rightarrow$ $\vec{K}_1, \vec{K}_2$
λ_2 mult. 1 $\rightarrow$ 1 row of zeros $\rightarrow$ $\vec{K} \rightarrow \vec{P}$

• 1 eigenvalue mult. 3 $\vec{K} \rightarrow \vec{P} \rightarrow \vec{Q}$ (probably)

As shown in the diagram, two large connected mixing tanks A and B initially contain 100 gallons of brine. Liquid is pumped in and out of the tanks as indicated in the figure; the mixture pumped between and out of a tank is assumed to be well-stirred. Construct a mathematical model in the form of a linear system of first-order differential equations for the number of pounds $x_1(t)$ and $x_2(t)$ of salt in tanks A and B, respectively, at time t. Write the system in matrix form.

$$\frac{dX}{dt} = \text{rate in} - \text{rate out}$$



$$\frac{dx_1}{dt} = \left(0 + \frac{x_2}{100}\right) - \left(\frac{x_1}{100} + \frac{2x_1}{100}\right)$$

$$\frac{dx_1}{dt} = \frac{1}{100}x_2 - \frac{3}{100}x_1$$

$$\frac{dx_2}{dt} = \frac{2x_1}{100} - \left(\frac{x_2}{100} + \frac{x_2}{100}\right)$$

$$\frac{dx_2}{dt} = \frac{1}{50}x_1 - \frac{1}{50}x_2$$

$$\vec{X}' = \begin{pmatrix} -\frac{3}{100} & \frac{1}{100} \\ \frac{1}{50} & -\frac{1}{50} \end{pmatrix} \vec{X}$$

AI response to where this model is used: This type of coupled mixing tank model shows up in several real engineering and scientific contexts:

Environmental & Water Treatment The most direct application is **wastewater treatment plants**, where tanks are linked in series or loops to progressively dilute or react contaminants. The exact flow-in/flow-out balance you see here models how pollutant concentrations evolve through each stage.

Pharmacokinetics (Drug Modeling) This is probably the *most important* real use. The human body is modeled as a series of "compartments" — blood plasma, tissue, organs — and a drug moves between them at known rates. The math is identical to these tank equations. Pharmacologists use this to predict:

- How long a drug stays in your system
- When blood concentration peaks
- Proper dosing intervals
- Chemical Reactors

In industrial chemistry, **continuous stirred-tank reactors (CSTRs)** are literally this setup. Reactants flow in, products flow out, and intermediate products recirculate between vessels. Petroleum refining and pharmaceutical manufacturing use multi-tank CSTR networks constantly.

Groundwater Contamination Aquifers are modeled as interconnected "cells" exchanging water at known rates. When a contaminant spills, hydrologists use these equations to predict how the plume spreads between underground zones over time.

HVAC and Atmospheric Modeling The atmosphere itself is divided into compartments (troposphere, boundary layer, urban airshed) with pollutants like CO_2 or smog transferring between them — same mathematical structure.

The reason this model is so powerful is that **any system where a substance moves between well-mixed reservoirs at proportional rates** reduces to the same form of coupled linear ODEs. The tanks are just the most intuitive way to introduce the concept.